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Note

The solution of two problems on bound polysemy

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Abstract

A pair of graphs (G_1, G_2) on the same set of vertices V is called bound polysemic, if there is a poset $P = (V, \leqslant)$ such that for all $u, v \in V$ with $u \neq v$, uv is an edge of G_1 if and only if there is some $w \in V$ such that $u \leqslant w$ and $v \leqslant w$ and uv is an edge of G_2 if and only if there is some $w \in V$ such that $w \leqslant u$ and $w \leqslant v$. Solving two problems posed by Tanenbaum (Electron. J. Comb. 7 (2000) R43), we characterize the bound polysemic pairs for which the poset P is unique and we describe an algorithm to recognize bound polysemic pairs in $\mathcal{O}(|V|^3)$ time.

Keywords: Bound graph; Poset; Polysemy; Polysemic pair; Clique

1. Introduction

We consider finite simple graphs $G = (V, E)$ with vertex set V and edge set E. The (*closed*) *neighbourhood* of a vertex $u \in V$ in G is $N_G(u) = \{v \in V \mid uv \in E\}$ $(N_G[u] = \{u\} \cup N_G(u))$. A vertex $u \in V$ is *isolated*, if $N_G(u) = \emptyset$. An *independent set* of G is a set of pairwise non-adjacent vertices. A *clique* of G is the vertex set of a (not necessarily maximal) complete subgraph of G. A clique is *trivial*, if it contains just one vertex. A vertex is *simplicial*, if its neighbourhood is a clique. An *edge clique cover* of G is a collection $\mathscr{C}$ of non-trivial cliques such that for every edge $uv \in E$ some clique in $\mathscr{C}$ contains both vertices u and v.

We also consider finite reflexive posets $P = (V, \leqslant)$. For $u \in V$, the set of upper (lower) bounds is $V_{\geqslant}(u) = \{v \in V \mid u \leqslant v\}$ $(V_{\leqslant}(u) = \{v \in V \mid v \leqslant u\})$. The set of maximal (minimal) elements of P is $\mathrm{Max}(P) = \{u \in V \mid V_{\geqslant}(u) = \{u\}\}$ $(\mathrm{Min}(P) = \{u \in V \mid V_{\leqslant}(u) = \{u\}\})$. For further definitions we refer to [5].

In [9], Tanenbaum introduced the notion of *bound polysemy*. A pair (G_1, G_2) of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices V is called *bound polysemic*, if there exists a poset $P = (V, \leqslant)$ on the set V such that for all $u, v \in V$ with $u \neq v$, $uv \in E_1$ if and only if $V_{\geqslant}(u) \cap V_{\geqslant}(v) \neq \emptyset$, i.e. u and v have a common upper bound in P, and $uv \in E_2$ if and only if $V_{\leqslant}(u) \cap V_{\leqslant}(v) \neq \emptyset$, i.e. u and v have a common lower bound in P. The poset P is called a *realization* of (G_1, G_2). The graphs G_1 and G_2 are called the *upper bound graph* and the *lower bound graph* of P, respectively. Upper bound graphs were introduced by McMorris and Zaslavsky in [8] and have been studied by various authors (cf. [1–4,7] and also the survey [6]).

In the present paper, we will solve two problems posed by Tanenbaum at the end of [9]. In the next section, we review the main results of [8,9] and derive some of their consequences. In Section 3, we characterize bound polysemic pairs that have a unique realization. Finally, in Section 4 we give an $\mathcal{O}(|V|^3)$ time algorithm to recognize bound polysemic pairs.

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2. Some known results

The main result of Tanenbaum in [9] is an extension of the following characterization of upper bound graphs due to McMorris and Zaslavsky [8].

Theorem 2.1 (McMorris and Zaslavsky [8]). *A graph $G=(V,E)$ is an upper bound graph if and only if there exists an edge clique cover $\mathscr{C}=\{C_1,C_2,\ldots,C_r\}$ of G for which there exists a set of vertices $R=\{v_1,v_2,\ldots,v_r\}\subseteq V$ such that $v_i\in C_j$ if and only if $i=j$.*

The next lemma collects some useful observations about Theorem 2.1.

Lemma 2.2. *Let $G=(V,E)$ be an upper bound graph and let $\mathscr{C}$ and R be as in Theorem 2.1. Then*

(i) $C_i=N_G[v_i]$ for $1\leqslant i\leqslant r$,
(ii) *R is a maximal independent set of non-isolated, simplicial vertices and*
(iii) *$\mathscr{C}=\{N_G[v]\,|\,v$ is non-isolated and simplicial$\}$, i.e. $\mathscr{C}$ is unique.*

Proof. Let $1\leqslant i\leqslant r$. Clearly, $C_i\subseteq N_G[v_i]$. For contradiction we assume that $N_G[v_i]\nsubseteq C_i$. This implies that there is a vertex $u\notin C_i$ with $v_iu\in E$. Since $\mathscr{C}$ is an edge clique cover, $\{u,v_i\}\subseteq C_j$ for some $1\leqslant j\leqslant r$ with $j\neq i$ which yields the contradiction $v_i\in C_j$. This implies $N_G[v_i]\subseteq C_i$ and, consequently, (i) holds.

Since the cliques in $\mathscr{C}$ are non-trivial, the vertices in R are non-isolated and, by (i), simplicial. The non-isolated, simplicial vertices of G induce a subgraph all components of which are complete graphs. Let V' be the vertex set of one of these complete components. If $R\cap V'=\emptyset$, then $\mathscr{C}$ is no edge clique cover, since no clique in $\mathscr{C}$ contains the non-isolated vertices in V'. Hence $R\cap V'\neq\emptyset$. Since $N_G[u]=N_G[v]$ for $u,v\in V'$, the set R contains exactly one vertex of V'. Hence R is a maximal independent set of non-isolated, simplicial vertices which proves (ii). Now (i) and (ii) easily imply (iii). □

The next theorem is the main result of [9].

Theorem 2.3 (Tanenbaum [9]). *A pair of graphs (G_1,G_2) with $G_1=(V,E_1)$ and $G_2=(V,E_2)$ is bound polysemic if and only if there exist edge clique covers $\mathscr{C}_1=\{C_{1,1},C_{1,2},\ldots C_{1,r}\}$ of G_1 and $\mathscr{C}_2=\{C_{2,1},C_{2,2},\ldots C_{2,s}\}$ of G_2 for which there exist two disjoint sets of vertices $R_1=\{v_{1,1},v_{1,2},\ldots,v_{1,r}\}\subseteq V$ and $R_2=\{v_{2,1},v_{2,2},\ldots,v_{2,s}\}\subseteq V$ such that*

(a) *$v_{1,i}\in C_{1,j}$ if and only if $i=j$,*
(b) *$v_{2,i}\in C_{2,j}$ if and only if $i=j$,*
(c) *$\bigcup_{i=1}^{r}C_{1,i}=\bigcup_{j=1}^{s}C_{2,j}$, and*
(d) *$C_{1,i}\cap C_{2,j}\neq\emptyset$ if and only if $v_{1,i}\in C_{2,j}$ and $v_{2,j}\in C_{1,i}$.*

(Note that G_i, $\mathscr{C}_i$ and R_i are as G, $\mathscr{C}$ and R in Theorem 2.1 for $i=1,2$.)

3. Uniqueness of the realization

Our next lemma allows us to consider only graphs without isolated vertices. Remember that the graph $G-I$ arises from G by deleting the vertices in I.

Lemma 3.1. *Let (G_1,G_2) with $G_1=(V,E_1)$ and $G_2=(V,E_2)$ be bound polysemic with realization $P=(V,\leqslant)$. For $i=1,2$ let I_i be the set of isolated vertices of G_i.*

Then $I_1=I_2$, (G_1-I_1,G_2-I_2) is bound polysemic, and (G_1,G_2) has a unique realization if and only if (G_1-I_1,G_2-I_2) has a unique realization.

Proof. It is obvious that a vertex $u\in V$ is isolated in G_1 if and only if $V_{\geqslant}(u)=V_{\leqslant}(u)=\{u\}$ if and only if u is isolated in G_2. Hence $I_1=I_2$ and (G_1-I_1,G_2-I_2) is bound polysemic with realization $(V\setminus I_1,\leqslant)$.

If $P=(V,\leqslant)$ and $P'=(V\setminus I_1,\leqslant')$ are posets such that $u\leqslant v$ if and only if $u\leqslant' v$ for all $u,v\in V\setminus I_1$ and $V_{\geqslant}(u)=V_{\leqslant}(u)=\{u\}$ for all $u\in I_1$, then P is a realization of (G_1,G_2) if and only if P' is a realization of (G_1-I_1,G_2-I_2). This implies the desired result. □

We proceed to the main result of this section.

Theorem 3.2. *Let* (G_1, G_2) *be bound polysemic such that neither* $G_1 = (V, E_1)$ *nor* $G_2 = (V, E_2)$ *has isolated vertices. Let* $\mathscr{C}_1$, $\mathscr{C}_2$, R_1 *and* R_2 *as in Theorem* 2.3. *For* $u \in V$ *and* $i = 1, 2$ *let* $R_i(u) = N_{G_i}[u] \cap R_i$.

Then (G_1, G_2) *has a unique realization if and only if the following conditions hold.*

(i) *If* $R_1(u) = \{v_{1,i}\}$ *for* $u \in V \setminus R_1$ *and* $1 \leqslant i \leqslant r$, *then there exists a vertex* $w \in V \setminus \{u, v_{1,i}\}$ *such that* $wv_{1,i} \in E_1$ *and* $wu \notin E_2$.

(ii) *If* $R_2(u) = \{v_{2,i}\}$ *for* $u \in V \setminus R_2$ *and* $1 \leqslant i \leqslant s$, *then there exists a vertex* $w \in V \setminus \{u, v_{2,i}\}$ *such that* $wv_{2,i} \in E_2$ *and* $wu \notin E_1$.

(iii) *Either* $R_1(v) \nsubseteq R_1(u)$ *or* $R_2(u) \nsubseteq R_2(v)$ *for* $u, v \in V \setminus (R_1 \cup R_2)$ *with* $u \neq v$.

Proof. Note that $R_i(u) \neq \emptyset$ for $u \in V$ and $i = 1, 2$, since the graphs G_1 and G_2 have no isolated vertices.

First, we assume that (G_1, G_2) has a unique realization and prove that the Conditions (i), (ii) and (iii) hold. Let the poset $P = (V, \leqslant)$ be such that $u \leqslant v$ for $u, v \in V$ if and only if either $v = v_{1,i}$ and $u \in C_{1,i}$ for some $1 \leqslant i \leqslant r$ or $u = v_{2,j}$ and $v \in C_{2,j}$ for some $1 \leqslant j \leqslant s$. It is easy to verify (cf. e.g. [9]) that P is a realization of (G_1, G_2) such that $R_1 = \mathrm{Max}(P)$, $R_2 = \mathrm{Min}(P)$, $R_1(u) = \{v \in \mathrm{Max}(P) \mid u \leqslant v\}$ and $R_2(u) = \{v \in \mathrm{Min}(P) \mid v \leqslant u\}$ for all $u \in V$. By assumption, P is the unique realization of (G_1, G_2). (Note that, by Lemma 2.2(iii), $\mathscr{C}_1$ and $\mathscr{C}_2$ are unique but that the poset P depends on the choice of the sets R_1 and R_2.)

In order to prove that Conditions (i) and (ii) hold, we assume, for contradiction, that $u \in V \setminus R_1$ and $1 \leqslant i \leqslant r$ are such that $R_1(u) = \{v_{1,i}\}$ and $wu \in E_2$ for all $w \in V \setminus \{u, v_{1,i}\}$ with $wv_{1,i} \in E_1$. Let $v'_{1,i} = u$ and $v'_{1,l} = v_{1,l}$ for $1 \leqslant l \leqslant r$ with $i \neq l$ and let

$$R'_1 = \{v'_{1,1}, v'_{1,2}, \dots v'_{1,r}\} = (R_1 \setminus \{v_{1,i}\}) \cup \{u\}.$$

We verify Conditions (a)–(d) in Theorem 2.3 for $\mathscr{C}_1$, $\mathscr{C}_2$, R'_1 and R_2. Since $R_1(u) = \{v_{1,i}\}$, we have, by Lemma 2.2(i), that

$$u \in N_{G_1}[v_{1,i}] \Bigg\backslash \bigcup_{\substack{1 \leqslant l \leqslant r \\ l \neq i}} N_{G_1}[v_{1,l}] = C_{1,i} \Bigg\backslash \bigcup_{\substack{1 \leqslant l \leqslant r \\ l \neq i}} C_{1,l}$$

and thus (a) holds. Conditions (b) and (c) are obvious, since they do not depend on R'_1. To verify Condition (d) we have to prove that $C_{1,i} \cap C_{2,j} \neq \emptyset$ for some $1 \leqslant j \leqslant s$ implies $u \in C_{2,j}$. Let $C_{1,i} \cap C_{2,j} \neq \emptyset$ for some $1 \leqslant j \leqslant s$. We obtain $v_{2,j} \in C_{1,i} = N_{G_1}[v_{1,i}]$ and thus $v_{2,j}v_{1,i} \in E_1$. By assumption, $v_{2,j}u \in E_2$ and thus $u \in C_{2,j} = N_{G_2}[v_{2,j}]$. Hence Conditions (a)–(d) in Theorem 2.3 are satisfied and there is a realization P' of (G_1, G_2) (defined similarly as P) such that $\mathrm{Max}(P') = R'_1 \neq R_1 = \mathrm{Max}(P)$, which is a contradiction. This implies that (i) holds. By symmetry, also (ii) holds.

In order to prove that Condition (iii) holds, we assume, for contradiction, that $R_1(v) \subseteq R_1(u)$ and $R_2(u) \subseteq R_2(v)$ for some $u, v \in V \setminus (R_1 \cup R_2)$ with $u \neq v$. Note that u and v are incomparable in P by the definition of P.

Let the poset $P'' = (V, \leqslant'')$ be such that for $x, y \in V$, $x \leqslant'' y$ if and only if $x \leqslant y$ or $u = x$ and $v = y$. That P'' is indeed a poset follows from the definition of P. It is straightforward to check that P'' is a realization of (G_1, G_2) that is different from P, which is a contradiction. Hence, Condition (iii) holds and the first part of the proof is complete.

Now, we assume that Conditions (i)–(iii) hold and prove that (G_1, G_2) has a unique realization. Let $P = (V, \leqslant)$ be a realization of (G_1, G_2).

First, we assume that $\mathrm{Max}(P) \neq R_1$. If $\mathrm{Max}(P) \setminus R_1 \neq \emptyset$, then let $u \in \mathrm{Max}(P) \setminus R_1$. We obtain that $N_{G_1}[u] = V_{\leqslant}(u)$ is a clique and that u is simplicial. By Lemma 2.2(iii), $N_{G_1}[u] \in \mathscr{C}$. This implies that $N_{G_1}[u] = C_{1,i}$ for some $1 \leqslant i \leqslant r$ with $v_{1,i} \neq u$ and that $R_1(u) = \{v_{1,i}\}$. Condition (i) implies that there is some $w \in V \setminus \{u, v_{1,i}\}$ such that $wv_{1,i} \in E_1$ and $wu \notin E_2$. We obtain that $w \in C_{1,i} = N_{G_1}[v_{1,i}] = N_{G_1}[u]$ which implies $w \leqslant u$. Hence $wu \in E_2$, which is a contradiction. Therefore, $\mathrm{Max}(P) \subseteq R_1$.

If $R_1 \setminus \mathrm{Max}(P) \neq \emptyset$, then let $u \in R_1 \setminus \mathrm{Max}(P)$. There is some $v \in \mathrm{Max}(P) \subseteq R_1$ such that $u \leqslant v$, which implies a contradiction to the independence of R_1 (cf. Lemma 2.2(ii)).

We obtain $\mathrm{Max}(P) = R_1$ and, by symmetry, $\mathrm{Min}(P) = R_2$. Furthermore, $C_{1,i} = V_{\leqslant}(v_{1,i})$ for $1 \leqslant i \leqslant r$ and $C_{2,j} = V_{\geqslant}(v_{2,j})$ for $1 \leqslant j \leqslant s$. This determines all relations in P which involve an element of $\mathrm{Max}(P) \cup \mathrm{Min}(P)$.

Now, let $u, v \in V \setminus (\mathrm{Max}(P) \cup \mathrm{Min}(P))$ with $u \neq v$. If $u \leqslant v$, then $R_1(v) \subseteq R_1(u)$ and $R_2(u) \subseteq R_2(v)$, which is a contradiction to (iii). Hence u and v are incomparable, P is uniquely determined and the proof is complete. □

In [7], McMorris and Myers proved the following analogous result for upper bound graphs.

Theorem 3.3 (McMorris and Myers [7]). *Let $G = (V, E)$ be an upper bound graph and let $\mathscr{C}$ and R be as in Theorem* 2.1. *For $u \in V$ let $R(u) = N_G[u] \cap R$.*

Then there is a unique poset $P = (V, \leqslant)$ such that G is the upper bound graph of P if and only if the following conditions hold.

(i) *$R(u) \neq R(v)$ for $u, v \in V$ with $u \neq v$.*
(ii) *$\{R(u) \mid u \in V \setminus R\}$ is an antichain with respect to inclusion.*

The reader can see that the conditions in Theorems 3.2 and 3.3 are quite different.

4. Recognition algorithm

In this section, we describe an algorithm that decides whether a given pair of graphs (G_1, G_2) with vertex set V is bound polysemic and that runs in time $\mathcal{O}(|V|^3)$. Note that this is exactly the same time complexity as Bergstrand and Jones's recognition algorithm for upper bound graphs in [1]. Lemma 3.1 easily implies that it is sufficient to consider pairs of graphs (G_1, G_2) such that neither G_1 nor G_2 has isolated vertices.

Algorithm 4.1.

Input: A pair (G_1, G_2) of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ without isolated vertices.

Output: "*Yes*" if (G_1, G_2) is bound polysemic, "*No*" otherwise. If the answer is "*Yes*", then the algorithm also produces edge clique covers $\mathscr{C}_1$ and $\mathscr{C}_2$ and sets R_1 and R_2 as in Theorem 2.3.

The algorithm executes the following five steps.

Step 1: Determine the sets of simplicial vertices S_1 of G_1 and S_2 of G_2.

Step 2: Determine the vertex sets $S_{1,1}, S_{1,2}, \ldots, S_{1,r}$ of the components of the subgraph $G_1[S_1]$ of G_1 induced by S_1. Determine the vertex sets $S_{2,1}, S_{2,2}, \ldots, S_{2,s}$ of the components of the subgraph $G_2[S_2]$ of G_2 induced by S_2.

Step 3: For $1 \leqslant i \leqslant r$ define $C_{1,i} := N_{G_1}[x]$ for an arbitrary vertex $x \in S_{1,i}$.
For $1 \leqslant j \leqslant s$ define $C_{2,j} := N_{G_2}[y]$ for an arbitrary vertex $y \in S_{2,j}$.
Define

$$\mathscr{C}_1 := \{C_{1,1}, C_{1,2}, \ldots, C_{1,r}\} \text{ and } \mathscr{C}_2 := \{C_{2,1}, C_{2,2}, \ldots, C_{2,s}\}.$$

If $\mathscr{C}_1$ is not an edge clique cover of G_1 or $\mathscr{C}_2$ is not an edge clique cover of G_2, then output "*No*". Otherwise, go to the next step.

Step 4: For $1 \leqslant i \leqslant r$ define

$$\tilde{S}_{1,i} := \left(S_{1,i} \cap \bigcap_{j: C_{1,i} \cap C_{2,j} \neq \emptyset} C_{2,j} \right) \Bigg\backslash \bigcup_{j: C_{1,i} \cap C_{2,j} = \emptyset} C_{2,j}$$

and for $1 \leqslant j \leqslant s$ define

$$\tilde{S}_{2,j} := \left(S_{2,j} \cap \bigcap_{i: C_{1,i} \cap C_{2,j} \neq \emptyset} C_{1,i} \right) \Bigg\backslash \bigcup_{i: C_{1,i} \cap C_{2,j} = \emptyset} C_{1,i} .$$

Step 5: Determine whether there exist disjoint sets

$$R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,r}\} \text{ and } R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,s}\}$$

such that $v_{1,i} \in \tilde{S}_{1,i}$ for $1 \leqslant i \leqslant r$ and $v_{2,j} \in \tilde{S}_{2,j}$ for $1 \leqslant j \leqslant s$.

If no such sets exist, then output "*No*". Otherwise, let R_1 and R_2 be such sets, and output "*Yes*", $\mathscr{C}_1$, $\mathscr{C}_2$, R_1 and R_2. The algorithm terminates at this point.

Theorem 4.2. *Algorithm* 4.1 *works correctly and can be implemented to run in time $\mathcal{O}(|V|^3)$ where V is the common vertex set.*

Proof. Let (G_1, G_2) be as in Algorithm 4.1 and let $n = |V|$.

In Step 1 the sets S_1 and S_2 can be determined by checking the $\mathcal{O}(n^2)$ adjacencies in the neighbourhood of every vertex in G_1 and G_2. This can be done in $\mathcal{O}(n^3)$ time. Note that the components of $G_1[S_1]$ and $G_2[S_2]$ are complete graphs. Therefore, determining the sets $S_{1,i}$ for $1 \leqslant i \leqslant r$ and $S_{2,j}$ for $1 \leqslant j \leqslant s$ in Step 2 can be done in $\mathcal{O}(n^2)$ time. By definition, $N_{G_1}[x] = N_{G_1}[x']$ for all $x, x' \in S_{1,i}$ with $1 \leqslant i \leqslant r$ and $N_{G_2}[x] = N_{G_2}[x']$ for all $x, x' \in S_{2,j}$ with $1 \leqslant j \leqslant s$. Hence,

$$\mathscr{C}_1 = \{N_{G_1}[x] \mid x \text{ is non-isolated and simplicial in } G_1\}$$

and

$$\mathscr{C}_2 = \{N_{G_2}[y] \mid y \text{ is non-isolated and simplicial in } G_2\}.$$

If (G_1, G_2) is bound polysemic, then, by Lemma 2.2(iii) and Theorem 2.3, $\mathscr{C}_1$ and $\mathscr{C}_2$ are edge clique covers of G_1 and G_2, respectively. Therefore, if the algorithm outputs "*No*" in Step 3, then the answer is correct.

Since G_1 and G_2 have no isolated vertices,

$$V = \bigcup_{i=1}^{r} C_{1,i} = \bigcup_{j=1}^{s} C_{2,j}$$

and Condition (c) in Theorem 2.3 is satisfied. It is straightforward to see that Step 3 and also Step 4 can be done in $\mathcal{O}(n^3)$ time.

Note that, if there are sets $R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,r}\}$ and $R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,s}\}$ such that $\mathscr{C}_1$, $\mathscr{C}_2$, R_1 and R_2 are as in the statement of Theorem 2.3, then $v_{1,i} \in C_{2,j}$ for some $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ implies that $v_{1,i} \in C_{1,i} \cap C_{2,j} \neq \emptyset$. Therefore, by symmetry, we can actually strengthen Condition (d) of Theorem 2.3 in the following way: for $1 \leqslant i \leqslant r$ and $1 \leqslant j \leqslant s$ we have that

$$C_{1,i} \cap C_{2,j} \neq \emptyset \quad \Leftrightarrow \quad v_{1,i} \in C_{2,j} \quad \Leftrightarrow \quad v_{2,j} \in C_{1,i}.$$

Hence $v_{1,i} \in \tilde{S}_{1,i}$ for $1 \leqslant i \leqslant r$ and $v_{2,j} \in \tilde{S}_{2,j}$ for $1 \leqslant j \leqslant s$.

On the other hand, if R_1 and R_2 are as in Step 5, i.e. R_1 and R_2 are disjoint, $R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,r}\}$ and $R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,s}\}$ such that $v_{1,i} \in \tilde{S}_{1,i}$ for $1 \leqslant i \leqslant r$ and $v_{2,j} \in \tilde{S}_{2,j}$ for $1 \leqslant j \leqslant s$, then Conditions (a), (b) and (d) of Theorem 2.3 are satisfied.

We will show that the problem of deciding whether sets R_1 and R_2 as in Step 5 exist and of finding them, if they exist, is equivalent to a matching problem in a bipartite graph. Let

$$U = \bigcup_{i=1}^{r} \tilde{S}_{1,i} \cup \bigcup_{j=1}^{s} \tilde{S}_{2,j}$$

and let $H = (V(H), E(H))$ be the bipartite graph with vertex set

$$V(H) = U \cup \{\tilde{S}_{1,i} \mid 1 \leqslant i \leqslant r\} \cup \{\tilde{S}_{2,j} \mid 1 \leqslant j \leqslant s\}$$

such that $uv \in E(H)$ if and only if $u \in U$, $v \in V(H) \setminus U$ and $u \in v$. Obviously, the sets R_1 and R_2 as in Step 5 exist if and only if the graph H has a matching M such that for each $v \in V(H) \setminus U$ there exists an edge in M that is incident to v. We have $|V(H)| \leqslant n + s + t \leqslant 3n$. Hence, using standard algorithms (cf. e.g. [5]), Step 5 can be done in $\mathcal{O}(n^3)$ time.

By Theorem 2.3 and the above remarks, the proof is complete. □

Finally, we want to point out that Theorems 3.2 and 4.2 imply that it is also possible to decide in $\mathcal{O}(|V|^3)$ time whether some bound polysemic pair (G_1, G_2) with vertex set V has a unique realization. Firstly, we can apply Algorithm 4.1 to obtain $\mathscr{C}_1$, $\mathscr{C}_2$, R_1 and R_2 as in Theorem 2.3 in time $\mathcal{O}(|V|^3)$. Secondly, it is easy to see that the (necessary and sufficient) Conditions (i)–(iii) in Theorem 3.2 can be checked in time $\mathcal{O}(|V|^3)$.

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