Available online at www.sciencedirect.com
SCIENCE DIRECT®

Discrete Mathematics 282 (2004) 251–255

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

# Competition polysemy

Miranca Fischermann[a], Werner Knoben[b], Dirk Kremer[a], Dieter Rautenbach[c]

[a] *Lehrstuhl II für Mathematik, RWTH-Aachen, 52062 Aachen, Germany*
[b] *Am Schürkamp 21, 46509 Xanten, Germany*
[c] *Forschungsinstitut für Diskrete Mathematik, Lennéstrasse 2, 53113 Bonn, Germany*

Received 31 May 2002; received in revised form 22 September 2003; accepted 26 November 2003

**Abstract**

Following a suggestion of Tanenbaum (Electron. J. Combin. 7 (2000) R43) we introduce the notion of competition polysemic pairs of graphs. A pair of (simple) graphs $(G_1, G_2)$ on the same set of vertices $V$ is called competition polysemic, if there exists a digraph $D = (V, A)$ such that for all $u, v \in V$ with $u \neq v$, $uv$ is an edge of $G_1$ if and only if there is some $w \in V$ such that $\overrightarrow{uw} \in A$ and $\overrightarrow{vw} \in A$ and $uv$ is an edge of $G_2$ if and only if there is some $w \in V$ such that $\overrightarrow{wu} \in A$ and $\overrightarrow{wv} \in A$. Our main results are a characterization of competition polysemic pairs $(G_1, G_2)$ in terms of edge clique covers of $G_1$ and $G_2$ and a characterization of the connected graphs $G$ for which there exists a tree $T$ such that $(G, T)$ is competition polysemic.
© 2004 Elsevier B.V. All rights reserved.

*Keywords:* Competition graph; Common enemy graph; Clique; Polysemic pair

## 1. Introduction

We consider finite simple graphs $G = (V, E)$ with vertex set $V$ and edge set $E$. A *clique* of $G$ is the vertex set of a (not necessarily maximal) complete subgraph of $G$. An *edge clique cover* of $G$ is a collection $\mathscr{C}$ of cliques such that for every edge $uv \in E$ some clique in $\mathscr{C}$ contains both vertices $u$ and $v$. A *block* of $G = (V, E)$ is a maximal 2-connected subgraph of $G$ and a vertex $u \in V$ for which $G - \{u\} = G[V \setminus \{u\}]$ has more components than $G$ is a *cutvertex*.

We also consider finite digraphs $D = (V, A)$ with vertex set $V$ and arc set $A$ which may contain loops but no multiple arcs. An arc in $D$ from $u$ to $v$ will be denoted by $\overrightarrow{uv}$ and the positive (negative) neighbourhood of a vertex $v \in V$ is $N_D^+(u) = \{v \in V \mid \overrightarrow{uv} \in A\}$ $(N_D^-(u) = \{v \in V \mid \overrightarrow{vu} \in A\})$. For further definitions we refer to [3].

In [11] Tanenbaum introduced the notion of *bound polysemy*. He called a pair $(G_1, G_2)$ of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices $V$ *bound polysemic*, if there exists a reflexive poset $P = (V, \leqslant)$ on the set $V$ such that for all $u, v \in V$ with $u \neq v$, $uv \in E_1$ if and only if there is some $w \in V$ such that $u \leqslant w$ and $v \leqslant w$ and $uv \in E_2$ if and only if there is some $w \in V$ such that $w \leqslant u$ and $w \leqslant v$.

In this situation the graphs $G_1$ and $G_2$ are called the *upper bound graph* and the *lower bound graph* of $P$, respectively. Upper bound graphs were introduced by McMorris and Zaslavsky in [7] (cf. also the survey [6]).

At the end of [11] Tanenbaum poses the problem of generalizing bound polysemy to *competition polysemy* using digraphs instead of posets. We will do so in the present paper. Consequently, we call a pair $(G_1, G_2)$ of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on a common set of vertices $V$ *competition polysemic*, if there exists a digraph $D = (V, A)$ on the same

*E-mail addresses:* miranca.fischermann@amv.de (M. Fischermann), wk@werner-knoben.de (W. Knoben), kremer@math2.rwth-aachen.de (D. Kremer), rauten@or.uni-bonn.de (D. Rautenbach).

0012-365X/$ - see front matter © 2004 Elsevier B.V. All rights reserved.
doi:10.1016/j.disc.2003.11.014

set of vertices such that for all $u, v \in V$ with $u \neq v$, $uv \in E_1$ if and only if $N_D^+(u) \cap N_D^+(v) \neq \emptyset$ and $uv \in E_2$ if and only if $N_D^-(u) \cap N_D^-(v) \neq \emptyset$.

In this situation $D$ is called a *realization* of $(G_1, G_2)$. Furthermore, the graphs $G_1$ and $G_2$ are called the *competition graph* and *common enemy graph* of $D$, respectively. Competition graphs were introduced by Cohen [1] to study food web models in ecology and have been studied by various authors (cf. eg. [2,4,8–10]).

Since every poset $P = (V, \leqslant)$ corresponds to a digraph $D = (V, A)$ such that $u \leqslant v$ for $u, v \in V$ if and only if $\overrightarrow{uv} \in A$, a pair of graphs is bound polysemic only if it is competition polysemic. In this sense competition polysemy generalizes bound polysemy. An unlabeled version of competition polysemy was studied in [5] (see also the corresponding comments in [11]).

In the next section we prove a characterization of competition polysemic pairs. In Section three we consider special cases of competition polysemy and prove a characterization of the connected graphs $G$ for which there exists a tree $T$ such that $(G, T)$ is competition polysemic.

## 2. A characterization of competition polysemy

Tanenbaum derived his characterization of bound polysemic pairs of graphs (Theorem 15 in [11]) from the characterization of upper bound graphs due to McMorris and Zaslavsky [7]. We adopt the same approach and start with the following characterization of competition graphs due to Dutton and Brigham [2] (cf. also [4,9]).

**Theorem 2.1** (cf. Dutton and Brigham [2]). *A graph $G = (V, E)$ is the competition graph of some digraph if and only if there exists an edge clique cover $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ of $G$ with $p \leqslant |V|$.*

If $\mathscr{C} = \{C_1, C_2, \ldots, C_p\}$ is an edge clique cover of $G$ with $p \leqslant |V|$, then we can choose a set of $p$ different vertices $R = \{v_1, v_2, \ldots, v_p\} \subseteq V$. We call $R$ a *set of distinct representatives* of the cliques in $\mathscr{C}$. (Note that—par abus de langage—we do not require $v_i \in C_i$ for $1 \leqslant i \leqslant p$.)

We proceed to our main result in this section.

**Theorem 2.2.** *A pair $(G_1, G_2)$ of graphs with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is competition polysemic if and only if there exist edge clique covers $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,p}\}$ of $G_1$ and $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,q}\}$ of $G_2$ for which there exist sets of distinct representatives $R_1 = \{v_{1,1}, v_{1,2}, \ldots, v_{1,p}\}$ and $R_2 = \{v_{2,1}, v_{2,2}, \ldots, v_{2,q}\}$, i.e. $|R_1| = p, |R_2| = q \leqslant |V|$, such that*

(i) *$v_{2,i} \in C_{1,j}$ if and only if $v_{1,j} \in C_{2,i}$,*
(ii) *if $C_{1,i} \cap C_{1,j} \neq \emptyset$, then there is some $1 \leqslant l \leqslant q$ such that $v_{1,i}, v_{1,j} \in C_{2,l}$ and*
(iii) *if $C_{2,i} \cap C_{2,j} \neq \emptyset$, then there is some $1 \leqslant l \leqslant p$ such that $v_{2,i}, v_{2,j} \in C_{1,l}$.*

**Proof.** First, we assume that $(G_1, G_2)$ with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is competition polysemic with realization $D = (V, A)$ and prove the existence of $\mathscr{C}_1$, $\mathscr{C}_2$, $R_1$ and $R_2$ as in the statement of the theorem.

Let $V = \{v_1, v_2, \ldots, v_n\}$ and for $1 \leqslant i \leqslant n$ let $v_{1,i} = v_{2,i} = v_i$, $C_{1,i} = N_D^-(v_{1,i})$ and $C_{2,i} = N_D^+(v_{2,i})$. Clearly, $u, v \in C_{1,i} = N_D^-(v_{1,i})$ holds for $u, v \in V$ with $u \neq v$ and $1 \leqslant i \leqslant n$ if and only if $v_{1,i} \in N_D^+(u) \cap N_D^+(v)$ or equivalently $uv \in E_1$. This implies that $\mathscr{C}_1 = \{C_{1,1}, C_{1,2}, \ldots, C_{1,n}\}$ is an edge clique cover of $G_1$. By symmetry, $\mathscr{C}_2 = \{C_{2,1}, C_{2,2}, \ldots, C_{2,n}\}$ is an edge clique cover of $G_2$. Furthermore, $v_{2,i} \in C_{1,j} = N_D^-(v_{1,j})$ holds if and only if $v_{1,j} \in C_{2,i} = N_D^+(v_{2,i})$ which implies (i). Finally, if $C_{1,i} \cap C_{1,j} \neq \emptyset$, then there is some $1 \leqslant l \leqslant n$ such that $v_{2,l} \in C_{1,i} \cap C_{1,j} = N_D^-(v_{1,i}) \cap N_D^-(v_{1,j})$. This implies $v_{1,i}, v_{1,j} \in N_D^+(v_{2,l}) = C_{2,l}$ which implies (ii) and, by symmetry, also (iii). This completes the first part of the proof.

Now, let $(G_1, G_2)$ be a pair of graphs with $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ and let $\mathscr{C}_1$, $\mathscr{C}_2$, $R_1$ and $R_2$ be as in the statement of the theorem.

Let the digraph $D$ have vertex set $V$ and arc set $A = A_1 \cup A_2$ where

$$A_1 = \{\overrightarrow{uv_{1,i}} | u \in C_{1,i}, 1 \leqslant i \leqslant p\} \quad \text{and} \quad A_2 = \{\overrightarrow{v_{2,j}u} | u \in C_{2,j}, 1 \leqslant j \leqslant q\}.$$

We prove that $(G_1, G_2)$ is competition polysemic with realization $D$.

Let $uv \in E_1$ for $u, v \in V$ with $u \neq v$. Since $\mathscr{C}_1$ is an edge clique cover of $G_1$, there is some $1 \leqslant i \leqslant p$ such that $u, v \in C_{1,i}$. This implies that $\overrightarrow{uv_{1,i}}, \overrightarrow{vv_{1,i}} \in A_1$ and $v_{1,i} \in N_D^+(u) \cap N_D^+(v) \neq \emptyset$.

Now, let $x \in N_D^+(u) \cap N_D^+(v) \neq \emptyset$ for $u, v \in V$ with $u \neq v$. We have that $\overrightarrow{ux}, \overrightarrow{vx} \in A_1 \cup A_2$.

If $\overrightarrow{ux}, \overrightarrow{vx} \in A_1$, then $x = v_{1,i}$ and $u, v \in C_{1,i}$ for some $1 \leqslant i \leqslant p$. This implies that $uv \in E_1$. If $\overrightarrow{ux} \in A_1$ and $\overrightarrow{vx} \in A_2$, then $x = v_{1,i}$ and $u \in C_{1,i}$ for some $1 \leqslant i \leqslant p$ and $v = v_{2,j}$ and $x = v_{1,i} \in C_{2,j}$ for some $1 \leqslant i \leqslant q$. Condition (i) implies that $v = v_{2,j} \in C_{1,i}$. Thus, $u, v \in C_{1,i}$ which implies that $uv \in E_1$. Similarly, if $\overrightarrow{ux} \in A_2$ and $\overrightarrow{vx} \in A_1$ we obtain $uv \in E_1$. Finally, if

$\overrightarrow{ux}, \overrightarrow{vx} \in A_2$, then $u=v_{2,i}$, $x \in C_{2,i}$, $v=v_{2,j}$ and $x \in C_{2,j}$ for some $1 \leqslant i,j \leqslant q$ with $i \neq j$. Since $x \in C_{2,i} \cap C_{2,j} \neq \emptyset$, Condition (iii) implies that there exists some $1 \leqslant l \leqslant p$ such that $v_{2,i}, v_{2,j} \in C_{1,l}$. Thus, $v_{2,i}v_{2,j} = uv \in E_1$. Hence, in all cases we have $uv \in E_1$.

We obtain that $uv \in E_1$ for $u,v \in V$ with $u \neq v$ if and only if $N_D^+(u) \cap N_D^+(v) \neq \emptyset$ which means that $G_1$ is the competition graph of $D$. By symmetry, $G_2$ is the common enemy graph of $D$ and hence $(G_1, G_2)$ is competition polysemic with realization $D$. This completes the proof. □

We want to point out that it is straightforward but tedious to derive Tanenbaum's characterization of bound polysemic pairs of graphs (Theorem 15 in [11]) from Theorem 2.2.

## 3. Special cases of competition polysemy

In Section 4 of [11] Tanenbaum investigates graphs $G$ such that $(G,H)$ is bound polysemic, and $H=G$ or $H$ is the complement $\bar{G}$ of $G$ or $H$ is a complete graph $K_n$ or $H$ is a tree. The analogous problems for competition polysemy are more complicated. For example, Tanenbaum shows that $(G,G)$ is bound polysemic if and only if the vertex set of $G$ is the disjoint union of cliques (cf. Theorem 8 in [11]). The following lemma shows that the graphs $G$ such that $(G,G)$ is competition polysemic cannot be characterized by forbidden induced subgraphs.

**Lemma 3.1.** *Let $G=(V_G,E_G)$ be a graph. There exists a graph $H=(V_H,E_H)$ of order at most $|E_G|$ such that $(G\cup H, G\cup H)$ is competition polysemic where $G \cup H = (V_G \cup V_H, E_G \cup E_H)$ and $V_G \cap V_H = \emptyset$.*

**Proof.** Let $\mathscr{C}=\{C_1,C_2,\ldots,C_p\}$ be an edge clique cover of $G=(V_G,E_G)$ such that $p$ is minimum. Since $\{\{u,v\} | uv \in E_G\}$ is an edge clique cover of $G$, we obtain that $p \leqslant |E_G|$. Let $D=(V_D,A_D)$ be the digraph with vertex set $V_D = V_G \cup \{v_1,v_2,\ldots,v_p\}$, where $V_G \cap \{v_1,v_2,\ldots,v_p\} = \emptyset$, and arc set

$$A_D = \bigcup_{i=1}^{p} \{\overrightarrow{wv_i}, \overrightarrow{v_iw} \mid w \in C_i\}.$$

Let $G_1 = (V_D, E_1)$ and $G_2 = (V_D, E_2)$ be the competition graph and common enemy graph of $D$, respectively. Since $N_D^+(v) = N_D^-(v)$ for every vertex $v \in V_D$, we have $G_1 = G_2$.

For $u,v \in V_G$ with $u \neq v$ we obtain that $uv \in E_G$ if and only if $u,v \in C_i$ for some $1 \leqslant i \leqslant p$ if and only if $v_i \in N_D^+(u) \cap N_D^+(v)$ if and only if $uv \in E_1$.

For $u \in V_G$ and $v \in \{v_1,v_2,\ldots,v_p\}$ we obtain that $N_D^+(u) \cap N_D^+(v) = \emptyset$ and hence $uv \notin E_1$. Let $H = (V_D \backslash V_G, E_1 \backslash E_G)$. Then, $H$ has $p \leqslant |E_G|$ vertices and $G_1 = G_2 = G \cup H$. This completes the proof. □

Another result of Tanenbaum is that $(G,\bar{G})$ is bound polysemic if and only if $G$ has just one vertex (cf. Theorem 10 in [11]). We will now present graphs $G$ of any order $n \geqslant 2$ such that $(G,\bar{G})$ is competition polysemic.

**Lemma 3.2.** *For $n \geqslant 2$ the pairs $(K_{1,n-1}, \bar{K}_{1,n-1})$ and $(\bar{K}_n, K_n)$ are competition polysemic where $K_{1,n-1}$ and $\bar{K}_n$ denote the star and the edgeless graph of order $n$, respectively.*

**Proof.** Let $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{v_1v_i \mid 2 \leqslant i \leqslant n\}$ and let $G=(V,E)$ and $H=(V,\emptyset)$. Clearly, $G \cong K_{1,n-1}$ and $H \cong \bar{K}_n$.

Furthermore, let $A_G = \{\overrightarrow{v_1v_i}, \overrightarrow{v_iv_i} \mid 2 \leqslant i \leqslant n\}$ and $A_H = \{\overrightarrow{v_1v_1}\} \cup \{\overrightarrow{v_1v_i},\ | 2 \leqslant i \leqslant n\}$. It is straightforward to verify that the pair $(G,\bar{G})$ is competition polysemic with realization $D_G = (V, A_G)$ and that the pair $(H,\bar{H})$ is competition polysemic with realization $D_H = (V, A_H)$. □

Tanenbaum shows that for any graph $G$ of order $n$ the pair $(G,K_n)$ is bound polysemic if and only if $G$ is an upper bound graph that contains a vertex of degree $n-1$ (cf. Theorem 11 in [11]). We have just seen in Lemma 3.2 that $(\bar{K}_n, K_n)$ is competition polysemic, which shows that the existence of a vertex of degree $n-1$ is not necessary for competition polysemy with $K_n$.

Our main result of this section generalizes Tanenbaum's characterization of graphs $G$ such that $(G,T)$ is bound polysemic for some tree $T$ in the case of connected graphs. Tanenbaum showed that $(G,T)$ is bound polysemic for some tree $T$ if and only if $G$ is complete and $T$ is a star (cf. Theorem 12 in [11]).

**Theorem 3.3.** *Let $G=(V,E_G)$ be a connected graph. There is a tree $T=(V,E_T)$ such that $(G,T)$ is competition polysemic if and only if*

(i) *at most one block of $G$ is not complete,*
(ii) *every cutvertex of $G$ lies in exactly two blocks of $G$ and*
(iii) *if some block of $G$ is not complete, then the vertex set of this block is the union of two cliques of $G$ that have exactly two common vertices and these vertices lie in no other block of $G$.*

**Proof.** First, we assume that $(G,T)$ is competition polysemic with realization $D$ where $G=(V,E_G)$ is a connected graph and $T=(V,E_T)$ is a tree.

Let $V=\{v_1,v_2,\ldots,v_n\}$ and for $1\leqslant i\leqslant n$ let $v_{1,i}=v_{2,i}=v_i$, $C_{1,i}=N_D^-(v_{1,i})$ and $C_{2,i}=N_D^+(v_{2,i})$. Let $\mathscr{C}_1=\{C_{1,1},C_{1,2},\ldots,C_{1,n}\}$ and $\mathscr{C}_2=\{C_{2,1},C_{2,2},\ldots,C_{2,n}\}$. As in the proof of Theorem 2.2 is follows that $\mathscr{C}_1$, $\mathscr{C}_2$, $R_1$ and $R_2$ are as in the statement of Theorem 2.2. (*Note that we use double indices '$1,i$' or '$2,j$' for vertices just in order to emphasize that a vertex corresponds to a certain clique in $\mathscr{C}_1$ or $\mathscr{C}_2$, respectively.*)

Since $T$ is a tree, $\mathscr{C}_2$ contains exactly $n-1$ different cliques of cardinality 2 and one clique that is a subset of one of the others. Without loss of generality let $C_{2,1}\subseteq C_{2,2}$.

If $v_{2,i}\in C_{1,j}\cap C_{1,k}\cap C_{1,l}$ for some $1\leqslant i\leqslant n$ and $1\leqslant j<k<l\leqslant n$, then $v_{1,j},v_{1,k},v_{1,l}\in C_{2,i}$, which implies a contradiction to $|C_{2,i}|\leqslant 2$. Hence, every vertex of $G$ lies in at most two cliques of $\mathscr{C}_1$. We denote this property of $G$ by $(*)$.

If $v_{2,s},v_{2,t}\in C_{1,i}\cap C_{1,j}$ for some $1\leqslant i<j\leqslant n$ and $1\leqslant s<t\leqslant n$, then $v_{1,i},v_{1,j}\in C_{2,s}\cap C_{2,t}$, which implies that $\{v_{1,i},v_{1,j}\}=C_{2,s}=C_{2,t}$ and hence $\{s,t\}=\{1,2\}$. Thus, for $1\leqslant i<j\leqslant n$ we obtain

$$|C_{1,i}\cap C_{1,j}|\leqslant 1,\quad \text{if } C_{2,1}\neq\{v_{1,i},v_{1,j}\}, \tag{1}$$

$$|C_{1,i}\cap C_{1,j}|=2,\quad \text{if } C_{2,1}=\{v_{1,i},v_{1,j}\}. \tag{2}$$

If $G$ contains a cycle that is not covered by a single clique in $\mathscr{C}_1$, then there are $t\geqslant 2$ cliques

$$C_{1,j_1},C_{1,j_2},\ldots,C_{1,j_t}\in\mathscr{C}_1$$

such that $C_{1,j_i}\neq C_{1,j_{i+1}}$ for every $1\leqslant i\leqslant t-1$ and $C_{1,j_t}\neq C_{1,j_1}$ and $t$ vertices

$$v_{f_1},v_{f_2},\ldots,v_{f_t}$$

such that $v_{f_i}\in C_{1,j_i}\cap C_{1,j_{i+1}}$ for every $1\leqslant i\leqslant t-1$ and $v_{f_t}\in C_{1,j_t}\cap C_{1,j_1}$ with $f_i\neq f_j$ for $i\neq j$.

We obtain, $v_{1,j_i},v_{1,j_{i+1}}\in C_{2,f_i}$ for every $1\leqslant i\leqslant t-1$ and $v_{1,j_t},v_{1,j_1}\in C_{2,f_t}$. Therefore $v_{1,j_i}v_{1,j_{i+1}}\in E_T$ for every $1\leqslant i\leqslant t-1$ and $v_{1,j_t}v_{1,j_1}\in E_T$. Since $T$ is a tree, we have $t=2$, $C_{2,f_1}=C_{2,f_2}=\{v_{1,j_1},v_{1,j_2}\}$ and $\{f_1,f_2\}=\{1,2\}$.

Hence, every cycle in $G$ that is not covered by a single clique in $\mathscr{C}_1$ is covered by the unique two cliques $C_{1,j_1},C_{1,j_2}$ with $C_{2,1}=C_{2,2}=\{v_{1,j_1},v_{1,j_2}\}$.

This implies that every clique $C_{1,i}$ with $v_{1,i}\notin C_{2,1}$ is the vertex set of a complete block in $G$. Furthermore, if some block $B$ of $G$ is not complete, then $C_{2,1}=C_{2,2}$ and $V(B)\subseteq C_{1,j_1}\cup C_{1,j_2}$ with $C_{2,1}=\{v_{1,j_1},v_{1,j_2}\}$. Since every block of $G$ which contains two vertices of a clique contains the whole clique, we obtain that $V(B)=C_{1,j_1}\cup C_{1,j_2}$. Thus, at most one block of $G$ is not complete and Condition (i) holds.

Since every cutvertex of $G$ lies in at least two blocks of $G$, we get, by $(*)$, that every cutvertex of $G$ lies in exactly two blocks of $G$ and Condition (ii) holds.

Now, let $G$ contain a block $B$ that is not complete. Then, $V(B)=C_{1,j_1}\cup C_{1,j_2}$ and $C_{2,1}=\{v_{1,j_1},v_{1,j_2}\}$. By (2), we obtain that $|C_{1,j_1}\cap C_{1,j_2}|=2$. By $(*)$, the two vertices in $C_{1,j_1}\cup C_{1,j_2}$ lie in no clique $C_{1,i}$ with $i\neq j_1,j_2$ and in no block of $G$ besides $B$. Hence Condition (iii) holds. This completes the first part of the proof.

Now, let $G=(V,E_G)$ be a connected graph such that the Conditions (i)–(iii) hold. Let $S$ be the set of cutvertices of $G$.

If one block of $G$ is not complete, then let this block be $B_0$, let $C_0$ and $C_1$ be two cliques of $G$ such that $V(B_0)=C_0\cup C_1$ and $|C_0\cap C_1|=2$. Let $\{x_0,x_1\}=C_0\cap C_1$ and define $N_i=C_i$ for $i=0,1$.

If all blocks of $G$ are complete, then let $x_0$ be an arbitrary vertex in $V\backslash S$, let $B_0$ be the unique block of $G$ that contains $x_0$, let $x_1=x_0$ and $N_i=V(B_0)$ for $i=0,1$.

It is straightforward to see that for $1\leqslant i\leqslant |S|$ we can (recursively) choose vertices $x_{i+1}\in S\backslash\{x_j\mid 2\leqslant j\leqslant i\}$ and define sets

$$N_{i+1}=\{x_{i+1}\}\cup\left(\{u\in V\mid ux_{i+1}\in E_G\}\backslash\bigcup_{j=0}^{i}N_j\right)$$

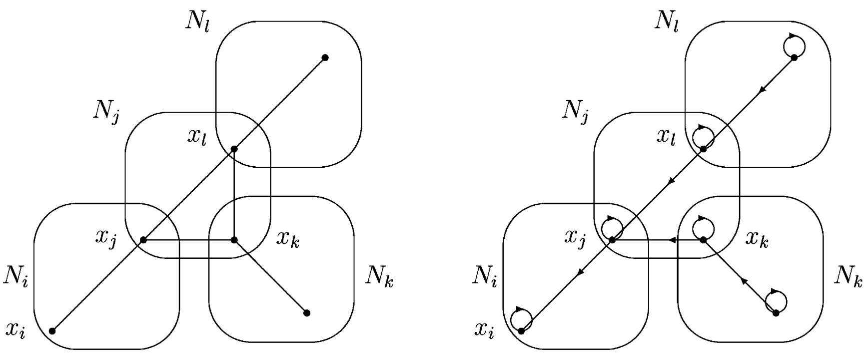


Fig. 1. $i < j < k < l$.

such that every set $N_i$ for $0 \leqslant i \leqslant |S| + 1$ is a clique of $G$ and if $i \geqslant 2$, then $N_i$ is the vertex set of a block in $G$. Furthermore, for $i \geqslant 2$ every cutvertex $x_i$ of $G$ lies in $N_i$ and $N_j$ for some unique $j < i$. (See the left part of Fig. 1 for illustration.)

Now, we define the digraph $D = (V, A)$ with vertex set $V$ and arc set

$$A = \{\overrightarrow{yx_j} \mid y \in N_j, 0 \leqslant j \leqslant |S| + 1\} \cup \{\overrightarrow{uu} | u \in V\}.$$

(See the right part of Fig. 1 for illustration.)

Let $E_1$ and $E_2$ be the edge sets of the competition graph and the common enemy graph of $D$, respectively. Note, that $N_D^+(x_0) = N_D^+(x_1) = \{x_0, x_1\}$ and for every $x \in V \setminus \{x_0, x_1\}$ we have $x \in N_i \setminus \{x_i\}$ and $N_D^+(x) = \{x, x_i\}$ for some $0 \leqslant i \leqslant |S| + 1$. Thus, for $u, v \in V$ with $u \neq v$ we obtain that $uv \in E_2$ if and only if $\{u, v\} = N_D^+(x)$ for some $x \in V$ if and only if $\{u, v\} = \{x, x_i\}$ and $x \in N_i \setminus \{x_i\}$ for some $0 \leqslant i \leqslant |S| + 1$. Hence, we obtain that $G_2 = (V, E_2)$ is a tree, since for every block $B$ of $G$ the subgraph $G_2[V(B)]$ induced by $V(B)$ in $G_2$ is a star, if $B$ is complete and a double star (=a tree of diameter 3), if $B = B_0$ and $B_0$ is not complete.

Now, it remains to prove that $G_1 = (V, E_1) = (V, E_G) = G$. Note that $N_D^-(x) = N_i$ if $x = x_i$ for $0 \leqslant i \leqslant |S| + 1$ and $N_D^-(x) = \{x\}$ if $x \in V \setminus \{x_0, x_1, \ldots, x_{|S|+1}\}$. Let $uv$ be an edge of $G$. If $uv \in E(B_0)$, then $u, v \in N_i$ for some $i \in \{0, 1\}$ which implies that $u, v \in N_D^-(x_i)$ for some $i \in \{0, 1\}$ and thus $uv \in E_1$. If $uv \in E(B)$ for some block $B \neq B_0$, then $B$ is complete and contains at least one cutvertex. If $i = \min\{2 \leqslant j \leqslant |S| \mid x_j \in V(B)\}$, then $u, v \in N_i = V(B)$ and $u, v \in N_D^-(x_i)$ which implies that $uv \in E_1$. This yields that $E_G \subseteq E_1$.

Conversely, let $uv \in E_1$. We have $u, v \in N_D^-(x)$ for some vertex $x \in V$ with $|N_D^-(x)| \geqslant 2$. This implies that $x = x_j$ and $u, v \in N_j$ for some $0 \leqslant j \leqslant |S| + 1$. Since $N_j$ is a clique in $G$, we obtain that $uv \in E_G$. Hence $E_G = E_1$ and the proof is complete. □

## References


[1] J.E. Cohen, Interval graphs and food webs: a finding and a problem, Rand Corporation Document 17696-PR, Santa Monica, CA, 1968.

[2] R.D. Dutton, R.C. Brigham, A characterization of competition graphs, Discrete Appl. Math. 6 (1983) 315–317.

[3] R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Vol. 1, Elsevier, North-Holland, Amsterdam, 1995.

[4] J.R. Lundgren, J.S. Maybee, A characterization of graphs of competition number $m$, Discrete Appl. Math. 6 (1983) 319–322.

[5] J.R. Lundgren, J.S. Maybee, F.R. McMorris, Two-graph inversion of competition graphs and bound graphs, Congr. Numer. 67 (1988) 136–144.

[6] T.A. McKee, F.R. McMorris, Topics in intersection graph theory, SIAM Monographs on Discrete Mathematics and Applications, Philadelphia, 1999.

[7] F.R. McMorris, T. Zaslavsky, Bound graphs of a partially ordered set, J. Combin. Inform. System Sci. 7 (1982) 134–138.

[8] F.S. Roberts, Food webs, competition graphs and the boxicity of ecological phase space, in: Theory and Applications of Graphs, Springer, New York, 1978, pp. 477–490.

[9] F.S. Roberts, J.E. Steif, A characterization of competition graphs of arbitrary digraphs, Discrete Appl. Math. 6 (1983) 323–326.

[10] D. Scott, The competition-common enemy graph of a digraph, Discrete Appl. Math. 17 (1987) 269–280.

[11] P.J. Tanenbaum, Bound graph polysemy, Electron. J. Combin. 7 (2000) R43, 12pp.